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On dual formulations of gravity

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Abstract: In this paper we consider a possibility to construct dual formulation of gravity where the main dynamical field is the connection and not that of tetrad $e_\mu{}^a$ or metric $g_{\mu\nu}$. Our approach is based on the usual dualization procedure which uses first order parent Lagrangians but in (Anti) de Sitter space and not in the flat Minkowski one. At first we consider dual formulation based on the usual terad formalism with Lorentz connection $\omega_\mu{}^{ab}$ as new dynamical field. It turns out that in $d = 3$ dimensions such dual formulation is related with the so called exotic parity-violating interactions for massless spin-2 particles. Then we construct a dual formulation of gravity where the main dynamical object is affine connection starting with the well known first order Palatini formulation but in (Anti) de Sitter background. The final result obtained by solving equations for the metric is the Lagrangian written by Eddington in his book in 1924. Also there is an interesting connection with attempts to construct gravitational analog of Born-Infeld electrodynamics.

Keywords: Classical Theories of Gravity, Field Theories in Higher Dimensions.

Contents

1. Introduction

Investigations of dual formulations for tensor fields are important for understanding of alternative formulations of known theories like gravity as well as understanding of their role in superstrings. In general, by dual formulation we mean any situation where the very same particle is described by different tensor fields. Common procedure for obtaining such dual formulations is based on the parent first order Lagrangians. As is well known in flat Minkowski space such dualization procedure leads to different results for massive and massless particles. At the same time in (Anti) de Sitter space-time gauge invariance requires introduction quadratic mass-like terms into the Lagrangians. As a result dualization for massless particles in (Anti) de Sitter spaces [1] goes exactly in the same way as that for massive particles [2] and gives results different from ones for dualization of massless particles [3] in flat Minkowski space.

In the first part of the paper paper using such dualization procedure we consider a possibility to construct dual formulation of gravity where the main dynamical quantity is a Lorentz connection field ${\omega_\mu}^{ab}$. Such a formulation was considered previously e.g. [4−9]. Also such dual formulation of $d = 3$ gravity was recently discussed in [10]. It turns out that in $d = 3$ dimensions such dual formulation of gravity is related with the so called exotic parity-violating interactions for massless spin-2 particles [11, 12]. So we start with $d = 3$ case and show that such exotic interaction can be viewed as higher derivatives interactions in terms Lorentz connection ${\omega_\mu}^{ab}$. Then we show how such interaction could be obtained from the usual gravitational interactions by dualization procedure starting with (Anti) de Sitter space and then considering a kind of flat limit. Then in the next section we consider straightforward generalization of such theory on arbitrary $d \geq 4$ dimensions.

But there exist another well known first order formalism for gravity usually called Palatini formalism, the main components being the metric and affine connection. Such formalism differs drastically from the tetrad one because affine connection is not a gauge invariant object (or, geometrically, it is not a covariant tensor) and does not have its own gauge invariance. In spite of this difference, as we are going to show in the second part of our paper, it is also possible to apply the same dualization procedure to obtain a formulation of gravity where the main dynamical field is the affine connection. Rather naturally and at the same time surprisingly the final result is nothing else but the Lagrangian written by Eddington in 1924 [13]! Also there is an interesting connection with attempts to construct gravitational analog of Born-Infeld electrodynamics [14 − 19].

2. Dual gravity in $d = 3$

Investigations of possible interactions for massless spin-2 particles have shown that in $d = 3$ case there exist non-trivial "exotic parity-violating" higher derivatives interactions [11, 12]. The simplest way to see this [20] is to start with the first order formulation for massless spin-2 particle using "triad" $h_\mu{}^a$ and Lorentz connection $\omega_\mu{}^{ab}$ and introduce dual variable $f_\mu{}^a = \frac{1}{2}\varepsilon^{abc}\omega_\mu{}^{bc}$. In this notations the Lagrangian for free massless spin-2 particles has a very simple form:

$$\mathcal{L}_0 = \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b - \varepsilon^{\mu\nu\alpha} f_\mu{}^a \partial_\nu h_\alpha{}^a \tag{2.1}$$

Here

$$\left\{ {}^{\mu\nu}_{ab} \right\} = \delta_a{}^\mu \delta_b{}^\nu - \delta_a{}^\nu \delta_b{}^\mu$$

and so on. This Lagrangian is invariant under the following local gauge transformations:

$$\delta h_{\mu a} = \partial_\mu \xi_a + \varepsilon_{\mu ab}\eta^b \qquad \delta f_\mu{}^a = \partial_\mu \eta^a \tag{2.2}$$

Then it is easy to check that if we add the following cubic terms to the Lagrangian and appropriate corrections to gauge transformation laws:

$$\mathcal{L}_1 = -\frac{K}{6}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} f_\mu{}^a f_\nu{}^b f_\alpha{}^c \qquad \delta_1 h_\mu{}^a = -K\varepsilon^{abc} f_\mu{}^b \eta^c \tag{2.3}$$

where K — arbitrary coupling constant, we obtain gauge invariant interacting theory. In this, equation of motion for the $f_\mu{}^a$ field are still algebraic, but non-linear now. So if we try to solve this equation in passing to second order formulation we get essentially non-linear theory with higher and higher derivatives terms. To see what kind of theory we get let us consider lowest order approximations. It will be convenient to introduce "dual torsion"

$$T^{\mu a} = -\varepsilon^{\mu\nu\alpha}\partial_\nu h_\alpha{}^a, \qquad \hat{T}^{\mu a} = T^{\mu a} - e^{\mu a} T$$

Then from the quadratic Lagrangian we easily obtain:

$$f_\mu^{(1)a} = \hat{T}^a{}_\mu, \qquad \mathcal{L}_0 = -\frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \hat{T}^a{}_\mu \hat{T}^b{}_\nu$$

In the next quadratic order we get:

$$f_\mu^{(2)a} = -K\hat{T}^a{}_b\hat{T}^b{}_\mu + \frac{K}{4}e_\mu{}^a[\hat{T}^b{}_c\hat{T}^c{}_b + \hat{T}^2]$$

Substituting this expressions back to the first order Lagrangian and keeping only terms cubic in fields we obtain an interactions in a first non-trivial order:

$$\mathcal{L}_1 = \frac{K}{6} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{T}^a{}_\mu \hat{T}^b{}_\nu \hat{T}^c{}_\alpha \tag{2.4}$$

and this is just the interaction considered in [11, 12]. Note here that such interactions do not necessarily violate parity because one can always assign $f_\mu{}^a$ to be a tensor, while $h_\mu{}^a$ — a pseudotensor. Now we can once again use a peculiarity of $d = 3$ space and dualize $h_\mu{}^a$ instead of $f_\mu{}^a$: $h_\alpha{}^a = \frac{1}{2}\varepsilon^{abc}\omega_\alpha{}^{bc}$ Then we can rewrite all results in terms of this new variable by noting that:

$$\hat{T}^a{}_\mu = (R_\mu{}^a - \frac{1}{4}\delta_\mu{}^a R) = \hat{R}_\mu{}^a$$

where we have introduced usual field strength:

$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab}, \qquad R_\mu{}^a = \delta^\nu{}_b R_{\mu\nu}{}^{ab}$$

Now the cubic interactions looks like:

$$\mathcal{L}_1 = \frac{K}{6} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{R}_\mu{}^a \hat{R}_\nu{}^b \hat{R}_\alpha{}^c \tag{2.5}$$

In $d = 3$ dimensions all these look just like trivial field redefinition, but looking this way it has to be clear that there should exist a generalization of such interactions on arbitrary $d \geq 4$. To see how this generalization could be constructed we have to reobtain the same results without use of peculiarities of $d = 3$ dimensions. Now we will show that it is indeed possible by following usual dualization procedure based on the parent first order Lagrangians. Crucial fact here is that dualization for massless particles in (Anti) de Sitter spaces [1] goes in way similar to the one for massive particles in flat space [2] and not to that for massless ones [3]. So let us return back to the free case and start with massless particle in (Anti) de Sitter background space. A first order Lagrangians looks now as follows:

$$\mathcal{L}_0 = \frac{1}{2} \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b - \varepsilon^{\mu\nu\alpha} f_\mu{}^a D_\nu h_\alpha{}^a + \frac{\kappa}{2} \left\{ {}^{\mu\nu}_{ab} \right\} h_\mu{}^a h_\nu{}^b \tag{2.6}$$

and is invariant under the following local gauge transformations:

$$\delta h_{\mu a} = D_\mu \xi_a + \varepsilon_{\mu ab}\eta^b \qquad \delta f_\mu{}^a = D_\mu \eta_a + \kappa \varepsilon_{\mu ab}\xi^b \tag{2.7}$$

Working with the first order formalism it is very convenient to use tetrad formulation of the underlying (Anti) de Sitter space. We denote tetrad as $e_\mu{}^a$ (let us stress that it is not a dynamical quantity here, just a background field) and Lorentz covariant derivative as D_μ. (Anti) de Sitter space is a constant curvature space with zero torsion, so we have:

$$D_{[\mu} e_{\nu]}{}^a = 0, \qquad [D_\mu, D_\nu] v^a = \kappa (e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a) v_b \tag{2.8}$$

where $\kappa = -2\Lambda/(d-1)(d-2)$.

Now we switch on usual gravitational interaction by adding to the Lagrangian the following cubic terms:

$$\mathcal{L}_1 = -\frac{k}{2} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} f_\mu{}^a f_\nu{}^b h_\alpha{}^c - \frac{k\kappa}{6} \left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{2.9}$$

as well as appropriate corrections to gauge transformation laws:

$$\delta_1 h_\mu{}^a = k\varepsilon^{abc}(f_\mu{}^b\xi^c + h_\mu{}^b\eta^c) \qquad \delta_1 f_\mu{}^a = k\varepsilon^{abc}(f_\mu{}^b\eta^c + \kappa h_\mu{}^b\xi^c) \tag{2.10}$$

Note that in $d = 3$ case this gives us complete interacting theory. Then we switch back to the usual variable: $f_\mu{}^a = \frac{1}{2}\varepsilon^{abc}\omega_\mu{}^{bc}$. Also in order to have canonical normalization of fields in dual theory (where ω is main dynamical quantity now, while h — just auxiliary field) we make a rescaling: $\omega \rightarrow \sqrt{\kappa}\omega$ and $h \rightarrow \frac{1}{\sqrt{\kappa}}h$. In this a quadratic Lagrangian takes the form:

$$\mathcal{L}_0 = \frac{\kappa}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} - \frac{1}{2}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \omega_\mu{}^{ab} D_\nu h_\alpha{}^c + \frac{1}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} h_\mu{}^a h_\nu{}^b \tag{2.11}$$

and gauge transformations leaving it invariant (now $\eta^a = \frac{1}{2}\varepsilon^{abc}\eta_{bc}$)

$$\delta h_{\mu a} = D_\mu \xi_a + \kappa\eta_{\mu a} \qquad \delta\omega_\mu{}^{ab} = D_\mu\eta^{ab} - e_\mu{}^a\xi^b + e_\mu{}^b\xi^a \tag{2.12}$$

At the same time an interacting Lagrangian in these variables looks like:

$$\mathcal{L}_1 = -\frac{k\sqrt{\kappa}}{2}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \omega_\mu{}^{ad}\omega_\nu{}^{bd} h_\alpha{}^c - \frac{k}{6\sqrt{\kappa}}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{2.13}$$

with appropriate corrections for gauge transformations:

$$\begin{aligned} \delta_1 h_\mu{}^a &= -k\sqrt{\kappa}(\omega_\mu{}^{ab}\xi^b + h_{\mu b}\eta^{ba}) \\ \delta_1\omega_\mu{}^{ab} &= -k\sqrt{\kappa}(\omega_\mu{}^{ac}\eta^{cb} - \omega_\mu{}^{bc}\eta^{ca}) + \frac{k}{\sqrt{\kappa}}(h_\mu{}^a\xi^b - h_\mu{}^b\xi^a) \end{aligned} \tag{2.14}$$

Usually in passing to the second order formulation one solves algebraic equation of motion for the ω field (which geometrically give zero torsion condition). Then putting results back into the initial first order Lagrangian one obtains ordinary second order formulation in terms of (symmetric) tensor field. Here we proceed another way and try to solve equation for h field which is also algebraic in (Anti) de Sitter background. This equation looks as:

$$\frac{\delta\mathcal{L}}{\delta h_\mu{}^a} = -\frac{1}{4}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} R_{\nu\alpha}{}^{bc} + \left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} h_\nu{}^b - \frac{k}{2\sqrt{\kappa}}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_\nu{}^b h_\alpha{}^c - \frac{k\sqrt{\kappa}}{2}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \omega_\nu{}^{bd}\omega_\alpha{}^{cd} \tag{2.15}$$

where now $R_{\mu\nu}{}^{ab} = D_\mu\omega_\nu{}^{ab} - D_\nu\omega_\mu{}^{ab}$. In the lowest order approximation we get:

$$h_\mu^{(1)a} = \hat{R}_\mu{}^a, \qquad \hat{R}_\mu{}^a = R_\mu{}^a - \frac{1}{4}e_\mu{}^a R \tag{2.16}$$

while a second order quadratic Lagrangian takes the form:

$$\mathcal{L}_0 = -\frac{1}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \hat{R}_\mu{}^a\hat{R}_\nu{}^b + \frac{\kappa}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} \tag{2.17}$$

Note, that appearance of quadratic mass-like terms is natural in (Anti) de Sitter background and does not mean that ω field becomes massive. It is important that besides usual gauge transformations $\delta\omega_\mu{}^{ab} = D_\mu\eta^{ab}$ this Lagrangian also invariant under the local shifts $\delta\omega_\mu{}^{ab} = -e_\mu{}^a\xi^b + e_\mu{}^b\xi^a$ which is a remnant of ξ-invariance of initial first order Lagrangian. To check this invariance one can use that under these transformations we have $\delta\hat{R}_\mu{}^a = D_\mu\xi^a$.

Now we proceed and consider next approximation with cubic interaction terms in the Lagrangian and linear terms in gauge transformation laws. Before we give explicit formulas let us discuss what kind of theory we obtain. Schematically the solution of h equation and cubic Lagrangian look like:

$$\begin{aligned} h^{(2)} &\sim \frac{k}{\sqrt{\kappa}}(D\omega)(D\omega) + k\sqrt{\kappa}\omega\omega \\ \mathcal{L}_1 &\sim \frac{k}{\sqrt{\kappa}}(D\omega)(D\omega)(D\omega) + k\sqrt{\kappa}(D\omega)\omega\omega \end{aligned} \tag{2.18}$$

So the "main" interaction terms are cubic three derivatives ones constructed from the gauge invariant field strengths $(D\omega)$, the coupling constant being $K = \frac{k}{\sqrt{\kappa}}$ and at this level theory is essentially abelian. Only the presence of nonzero cosmological term adds one derivative Yang-Mills type coupling with dimensionless coupling constant being $g = k\sqrt{\kappa}$. In this, our theory becomes non-abelian, the gauge group being the Lorentz group. The non-trivial interactions given above could be reproduced now in a kind of "flat" limit when $k \to 0$ and $\kappa \to 0$ keeping K fixed. Indeed, in this limit we obtain:

$$h_\mu^{(2)a} = K\left[-\hat{R}_\mu{}^b\hat{R}_b{}^a + \hat{R}_\mu{}^a\hat{R} + \frac{1}{4}e_\mu{}^a\hat{R}_b{}^c\hat{R}_c{}^b - \frac{1}{4}e_\mu{}^a\hat{R}^2\right] \tag{2.19}$$

while the cubic terms in the Lagrangian take the same simple form as before:

$$\mathcal{L}_1 = -\frac{K}{6}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \hat{R}_\mu{}^a\hat{R}_\nu{}^b\hat{R}_\alpha{}^c \tag{2.20}$$

Besides the trivial at these limit invariance under the η^{ab} gauge transformations this Lagrangian is also invariant under the local shifts ξ^a with appropriate corrections:

$$\delta_1\omega_\mu{}^{ab} = K(\hat{R}_\mu{}^a\xi^b - \hat{R}_\mu{}^b\xi^a)$$

Let us stress that it is the invariance under these shifts that fixes the particular structure of cubic interactions among many other possible gauge invariant terms that could be easily constructed.

3. Dual gravity in $d \geq 4$

In this section we consider straightforward generalization of the procedure given above to the case of arbitrary $d \geq 4$ space-times. Again we start with the first order formulation of massless spin-2 particle in (Anti) de Sitter background with the Lagrangian:

$$\mathcal{L}_0 = \frac{\kappa}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} - \frac{1}{2}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} D_\mu\omega_\nu{}^{ab}h_\alpha{}^c + \frac{d-2}{2}\left\{ \begin{smallmatrix} \mu\nu \\ ab \end{smallmatrix} \right\} h_\mu{}^a h_\nu{}^b \tag{3.1}$$

Here we have already made a rescaling of fields appropriate for dual version. Then we add the usual gravitational interactions at the first non-trivial (cubic) order:

$$\mathcal{L}_1 = \frac{k\sqrt{\kappa}}{2}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} \omega_\mu{}^{ad}\omega_\nu{}^{bd}h_\alpha{}^c - \frac{k}{4\sqrt{\kappa}}\left\{ \begin{smallmatrix} \mu\nu\alpha\beta \\ abcd \end{smallmatrix} \right\} D_\mu\omega_\nu{}^{ab}h_\alpha{}^c h_\beta{}^d + \frac{(2d-5)k}{6\sqrt{\kappa}}\left\{ \begin{smallmatrix} \mu\nu\alpha \\ abc \end{smallmatrix} \right\} h_\mu{}^a h_\nu{}^b h_\alpha{}^c \tag{3.2}$$

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As is well known working with tetrad formulation of gravity and especially with supergravity theories it is very convenient to use the so called "1 and 1/2" order formalism. But here to construct a dual theory we have to work in a "honest" first order formalism taking into account gauge transformations for all fields. In this approximation they have the following form:

$$
\begin{aligned}
\delta_1 \omega_\mu{}^{ab} &= \frac{k}{\sqrt{\kappa}}[\xi^\nu R_{\nu\mu}{}^{ab} + (R_\mu{}^a \xi^b - R_\mu{}^b \xi^a) + \frac{1}{d-2}\xi^\nu (e_\mu{}^a R_\nu{}^b - e_\mu{}^b R_\nu{}^a) - \\
&\quad - \frac{1}{2(d-2)}(e_\mu{}^a \xi^b - e_\mu{}^b \xi^a) R - (d-2)(h_\mu{}^a \xi^b - h_\mu{}^b \xi^a)] \qquad (3.3) \\
\delta_1 h_\mu{}^a &= k\sqrt{\kappa}\omega_\mu{}^{ab}\xi^b
\end{aligned}
$$

for the ξ^a-transformations as well as

$$
\delta_1 \omega_\mu{}^{ab} = k\sqrt{\kappa}(\omega_\mu{}^{ac}\eta^{cb} - \omega_\mu{}^{bc}\eta^{ca}) \qquad \delta_1 h_\mu{}^a = k\sqrt{\kappa} h_{\mu b}\eta^{ba} \tag{3.4}
$$

for the η^{ab}-ones. Note that the main difference from the $d = 3$ case is rather complicated form for the ξ-transformations of ω field. As we will see this leads to the essential difference in the structure of interacting Lagrangian. Now we try to solve algebraic equation for h field which in this approximation looks as follows:

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta h_\mu{}^a} &= -\frac{1}{4}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} R_{\nu\alpha}{}^{bc} + (d-2)\left\{ {}^{\mu\nu}_{ab} \right\} h_\nu{}^b - \frac{k}{4\sqrt{\kappa}}\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} R_{\nu\alpha}{}^{bc} h_\beta{}^d + \\
&\quad + \frac{(2d-5)k}{2\sqrt{\kappa}}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} h_\nu{}^b h_\alpha{}^c + \frac{k\sqrt{\kappa}}{2}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \omega_\nu{}^{bd}\omega_\alpha{}^{cd} \qquad (3.5)
\end{aligned}
$$

This equation is a non-linear one. Moreover, if one consider next to the linear approximations then one obtains even more non-linear terms. So it seems hardly possible to get general solution of this equation, but nothing prevent us from solving it iteratively, order by order. Here we restrict ourselves by the linear approximation as in the previous case. In the lowest order approximation we get:

$$
h_\mu^{(1)a} = \frac{1}{d-2}\hat{R}_\mu{}^a, \qquad \hat{R}_\mu{}^a = R_\mu{}^a - \frac{1}{2(d-1)} e_\mu{}^a R \tag{3.6}
$$

and in this notations the structure of quadratic second derivative Lagrangian looks very similar to the $d = 3$ case:

$$
\mathcal{L}_0 = -\frac{1}{2(d-2)}\left\{ {}^{\mu\nu}_{ab} \right\} \hat{R}_\mu{}^a \hat{R}_\nu{}^b + \frac{\kappa}{2}\left\{ {}^{\mu\nu}_{ab} \right\} \omega_\mu{}^{ac}\omega_\nu{}^{bc} \tag{3.7}
$$

The formulas in the next approximation could be greatly simplified if we introduce traceless conformal Weyl tensor:

$$
C_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - \frac{1}{d-2} e_{[\mu}{}^{[a} R_{\nu]}{}^{b]} + \frac{1}{(d-1)(d-2)} e_\mu{}^{[a} e_\nu{}^{b]} R \tag{3.8}
$$

in this, the following useful relation holds:

$$
R_{\mu\nu}{}^{ab} = C_{\mu\nu}{}^{ab} + \frac{1}{d-2} e_{[\mu}{}^{[a} \hat{R}_{\nu]}{}^{b]} \tag{3.9}
$$

As in the $d=3$ case it is possible to consider a "flat" limit with when $k \to 0$ and $\kappa \to 0$ keeping $K = \frac{k}{\sqrt{\kappa}}$ fixed. In this limit a solution of h equation in the next order gives:

$$h_\mu^{(2)a} = -\frac{K}{(d-2)^3}[(d-2)C_{\mu\nu}{}^{ab}\hat{R}_b{}^\nu - \hat{R}_\mu{}^\nu\hat{R}_\nu{}^a + \hat{R}_\mu{}^a\hat{R} + \frac{1}{2(d-1)}e_\mu{}^a[(\hat{R}\hat{R}) - \hat{R}^2]] \quad (3.10)$$

Then putting this expression back to the initial first order Lagrangian and keeping only cubic terms we obtain the following three derivatives Lagrangian:

$$\mathcal{L}_1 = -\frac{K}{2(d-2)^2}[\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} C_{\mu\nu}{}^{ab}\hat{R}_\alpha{}^c\hat{R}_\beta{}^d + \frac{d-4}{3(d-2)}\left\{ {}^{\mu\nu\alpha}_{abc} \right\} \hat{R}_\mu{}^a\hat{R}_\nu{}^b\hat{R}_\alpha{}^c] \quad (3.11)$$

Again this particular structure of the Lagrangian is fixed not only by the invariance under the usual gauge transformations $\delta\omega_\mu{}^{ab} = \partial_\mu\eta^{ab}$, but also by the invariance under the local ξ shifts with the linear terms being:

$$\delta_1\omega_\mu{}^{ab} = K[\xi^\nu C_{\nu\mu}{}^{ab} + \frac{1}{d-2}(\hat{R}_\mu{}^b\xi^a - \hat{R}_\mu{}^a\xi^b)] \quad (3.12)$$

The following identities turn out to be useful:

$$\begin{aligned} D_a C_{\mu\nu}{}^{ab} &= \frac{1}{d-2}(D_\mu\hat{R}_\nu{}^b - D_\nu\hat{R}_\mu{}^b) \\ D_a\hat{R}_\mu{}^a &= D_\mu\hat{R} \end{aligned} \quad (3.13)$$

Note, that the general structure of the Lagrangian obtained is in agreement with the $d=3$ case. Indeed, in $d=3$ conformal Weyl tensor is identically zero, so the first term is absent. It is interesting to note that the $d=4$ case is also special, because in this and only this case the second term is absent.

4. Metric and affine connection

Let us start with the first order Lagrangian describing free massless spin-2 particle in flat Minkowski space:

$$\mathcal{L}_0 = h^{\mu\nu}(\partial_\alpha\Gamma_{\mu\nu}{}^\alpha - \partial_\mu\Gamma_\nu) + \eta^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha\Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta\Gamma_{\nu\beta}{}^\alpha) \quad (4.1)$$

Here $h_{\mu\nu}$ is symmetric second rank tensor while $\Gamma_{\mu\nu}{}^\alpha$ is assumed to be symmetric on the lower pair of indices. We denote $\Gamma_\alpha = \Gamma_{\alpha\beta}{}^\beta$, $\Gamma^\alpha = \eta^{\mu\nu}\Gamma_{\mu\nu}{}^\alpha$ (note, that Γ_α and Γ^α are in general different objects). This Lagrangian is invariant under the following local gauge transformations:

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}(\partial\xi), \qquad \delta\Gamma_{\mu\nu}{}^\alpha = -\partial_\mu\partial_\nu\xi^\alpha \quad (4.2)$$

As is well known, if one solves the algebraic equation of motion for the Γ field and put the result back into the Lagrangian one obtains usual second order Lagrangian for the symmetric tensor $h_{\mu\nu}$. In order to have a possibility to construct dual formulation where the main dynamical object is Γ we move from the flat Minkowski space to (Anti) de Sitter space. Let $\bar{g}_{\mu\nu}$ be a metric for this space (it is not a dynamical quantity, just a background field here) and D_μ — derivatives covariant with respect to background connection winch is torsionless and metric compatible:

$$D_\alpha \bar{g}_{\mu\nu} = 0, \qquad [D_\mu, D_\nu] v_\alpha = \bar{R}_{\mu\nu,\alpha}{}^\beta(\bar{g}) v_\beta = \kappa(\bar{g}_{\mu\alpha}\delta_\nu{}^\beta - \delta_\mu{}^\beta \bar{g}_{\nu\alpha}) v_\beta \tag{4.3}$$

where $\kappa = -2\Lambda/(d-1)(d-2)$. First of all we have to replace in the Lagrangian as well as in the gauge transformations the flat metric $\eta_{\mu\nu}$ by $\bar{g}_{\mu\nu}$ and partial derivatives ∂_μ by covariant ones D_μ:

$$\begin{aligned} \mathcal{L}_0 &= h^{\mu\nu}(D_\alpha \Gamma_{\mu\nu}{}^\alpha - D_\mu \Gamma_\nu) + \bar{g}^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha \Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta \Gamma_{\nu\beta}{}^\alpha) \\ \delta h_{\mu\nu} &= D_\mu \xi_\nu + D_\nu \xi_\mu - \bar{g}_{\mu\nu}(D\xi), \qquad \delta \Gamma_{\mu\nu}{}^\alpha = -\frac{1}{2}(D_\mu D_\nu + D_\nu D_\mu)\xi^\alpha \end{aligned} \tag{4.4}$$

Now (just because covariant derivatives do not commute) our Lagrangian is not invariant under the gauge transformations. Indeed, simple calculations give:

$$\delta \mathcal{L}_0 = \kappa[(d-2)\Gamma_\mu \xi^\mu - \frac{d-3}{2}\Gamma^\mu \xi_\mu - \frac{3d-1}{2} h^{\mu\nu} D_\mu \xi_\nu + h(D\xi)]$$

But gauge invariance could be easily restored by adding terms quadratic in $h_{\mu\nu}$ field to the Lagrangian as well as appropriate corrections for the gauge transformations:

$$\begin{aligned} \Delta \mathcal{L}_0 &= \frac{\kappa(d-1)}{2}[h^{\mu\nu} h_{\mu\nu} - \frac{1}{d-2} h^2] \\ \delta' \Gamma_{\mu\nu}{}^\alpha &= \frac{\kappa}{2}(\delta_\mu{}^\alpha \xi_\nu + \delta_\nu{}^\alpha \xi_\mu) - \kappa \bar{g}_{\mu\nu} \xi^\alpha \end{aligned} \tag{4.5}$$

Now one can easily solve the equations for the $h_{\mu\nu}$ field, which are also algebraic now, to obtain:

$$h_{\mu\nu} = \frac{1}{\kappa(d-1)}[R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R] \tag{4.6}$$

where we introduced a symmetric second rank tensor (it is not a full Ricci tensor yet, only the first part of it):

$$R_{(\mu\nu)} = \frac{1}{2}(D_\mu \Gamma_\nu + D_\nu \Gamma_\mu) - D_\alpha \Gamma_{\mu\nu}{}^\alpha \tag{4.7}$$

Then if we put this expression back into the initial first order Lagrangian we obtain dual second order formulation for massless spin-2 particle in terms of Γ field:

$$\mathcal{L}_{II} = -\frac{1}{\kappa(d-1)}[R^{\mu\nu} R_{\mu\nu} - \frac{1}{2} R^2] + \bar{g}^{\mu\nu}(\Gamma_{\mu\nu}{}^\alpha \Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta \Gamma_{\nu\beta}{}^\alpha) \tag{4.8}$$

A natural question arises: our field $\Gamma_{\mu\nu}{}^\alpha$ has a lot of independent components (40 in $d = 4$ instead of two helicities for massless spin-2 particle), so there should exist a large gauge symmetry in such a model. And indeed, it is easy to check that the kinetic terms in our second order Lagrangian are invariant under the local "affine" transformations:

$$\begin{aligned} \delta \Gamma_{\mu\nu}{}^\alpha &= \partial_\mu z_\nu{}^\alpha + \partial_\nu z_\mu{}^\alpha + \frac{1}{d-1}[\delta_\mu{}^\alpha (\partial z)_\nu + \delta_\nu{}^\alpha (\partial z)_\mu] - \\ &- \frac{1}{d-1}[\delta_\mu{}^\alpha \partial_\nu z + \delta_\nu{}^\alpha \partial_\mu z] \end{aligned} \tag{4.9}$$

where $z_\mu{}^\nu$ is arbitrary second rank tensor and $z = z_\mu{}^\mu$.

Now, having in our disposal an alternative description for massless spin-2 particle, it is natural to see how an interaction in such dual theory looks like. Nice feature of Palatini formulation is that switching on an interaction is a simple one step procedure [21]. But as we have seen, it is very important for the possibility to construct dual formulations to work not in a flat Minkowski space but in (Anti) de Sitter space. So we start with the usual Lagrangian with the cosmological term:

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \Lambda \sqrt{-g} \tag{4.10}$$

where now

$$R_{\mu\nu} = \frac{1}{2}(D_\mu \Gamma_\nu + D_\nu \Gamma_\mu) - D_\alpha \Gamma_{\mu\nu}{}^\alpha + \Gamma_{\mu\nu}{}^\alpha \Gamma_\alpha - \Gamma_{\mu\alpha}{}^\beta \Gamma_{\nu\beta}{}^\alpha \tag{4.11}$$

Then we introduce a convenient combination $\hat{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and rewrite a Lagrangian as:

$$\mathcal{L} = \hat{g}^{\nu\nu} R_{\mu\nu} + \Lambda det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} \tag{4.12}$$

The crucial point here is that first term contains $\hat{g}$ only linearly. As a result it is possible to get complete nonlinear solution of the $\hat{g}$ equations. We obtain (up to some numerical coefficients):

$$\hat{g}^{\mu\nu} \simeq \sqrt{det(R_{\mu\nu})} (R^{\mu\nu})^{-1} \tag{4.13}$$

At last, if we put this expression back into the first order Lagrangian we obtain (again up to normalization) a very simple and elegant Lagrangian:

$$\mathcal{L} = \sqrt{det(R_{\mu\nu})} \tag{4.14}$$

And it is just a Lagrangian written by Eddington eighty years ago in his book [13]! This result is very natural because this Lagrangian is the only invariant that could be constructed out of the affine connection alone, without any use of metric or any other objects, but it is exiting that this Lagrangian turns out to be dual formulation of usual gravity theory. Let us stress once again that working in a flat Minkowski space it is very hard if at all possible to give any reasonable physical interpretation to such model. But let us consider this model on a (Anti) de Sitter background. For that purpose we represent a total connection as $\Gamma_{\mu\nu}{}^\alpha = \bar{\Gamma}_{\mu\nu}{}^\alpha + \tilde{\Gamma}_{\mu\nu}{}^\alpha$ where $\bar{\Gamma}_{\mu\nu}{}^\alpha$ is a background connection while $\tilde{\Gamma}_{\mu\nu}{}^\alpha$ — small perturbation around it (see e.g. [22, 23]). Then for the curvature tensor we will have:

$$R_{\mu\nu,\alpha}{}^\beta = \bar{R}_{\mu\nu,\alpha}{}^\beta + [D_\mu \tilde{\Gamma}_{\nu\alpha}{}^\beta + \tilde{\Gamma}_{\mu\alpha}{}^\rho \tilde{\Gamma}_{\rho\nu}{}^\beta - (\mu \leftrightarrow \nu)] \tag{4.15}$$

where $\bar{R}_{\mu\nu,\alpha}{}^\beta$ is a curvature tensor for the background connection and D_μ is a derivative covariant with respect to $\bar{\Gamma}$. Then for the constant curvature space we have $\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$ so the Lagrangian takes the form:

$$\mathcal{L} = \sqrt{det(\Lambda \bar{g}_{\mu\nu} + \tilde{R}_{\mu\nu})} \tag{4.16}$$

It is interesting that Lagrangians of such kind have already been investigated e.g. [14 − 19] in attempts to construct gravitational analog of the Born-Infeld electrodynamics. But now the interpretation of the Lagrangian is drastically different. Indeed, let us use the well known decomposition for the determinant

$$\sqrt{det(I+A)} = 1 + \frac{1}{2}Sp(A) + \frac{1}{8}(Sp(A))^2 - \frac{1}{4}Sp(A^2) + \cdots .$$

where A is any matrix. Then if we consider the curvature $R_{\mu\nu}$ as being expressed in terms of metric and its second derivatives the first linear terms gives scalar curvature while quadratic terms give higher derivative terms leading to the appearance of ghosts. But here the main dynamical quantity is affine connection Γ and the curvature $R_{\mu\nu}$ contains only first derivatives. As a result a term linear in R is just a total derivative and could be dropped out of the action, while the quadratic terms give exactly the kinetic terms we obtained above.

Finally, let us add some comments on possible interaction with matter in such formulation of gravity. The most clear and straightforward way to obtain these interactions is to start with usual interactions in first order form and then try to go to the dual formulation. For example, for the scalar field we get:

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}[g^{\mu\nu}R_{\mu\nu} + \Lambda + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2}{2}\varphi^2] = \\ &= \hat{g}^{\mu\nu}(R_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi) + (\Lambda - \frac{m^2}{2}\varphi^2)det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} \end{aligned} \tag{4.17}$$

and the second line shows that the main effect is the replacement of $R_{\mu\nu}$ by $R_{\mu\nu} + \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi$ (compare [19]). Also if scalar field has nonzero mass the cosmological constant Λ is replaced by field dependent combination $\Lambda - \frac{m^2}{2}\varphi^2$. But for the vector field (even massless) the situation turns out to be much more complicated because even for the minimal interaction:

$$\begin{aligned} \mathcal{L} &= \sqrt{-g}[g^{\mu\nu}R_{\mu\nu} + \Lambda - \frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}] = \\ &= \hat{g}^{\nu\nu}R_{\mu\nu} + \Lambda det(\hat{g}^{\mu\nu})^{\frac{1}{d-2}} - \frac{1}{4}det(\hat{g}^{\mu\nu})^{-\frac{1}{d-2}}\hat{g}^{\mu\alpha}\hat{g}^{\nu\alpha}F_{\mu\nu}F_{\alpha\beta} \end{aligned} \tag{4.18}$$

equations for the $\hat{g}$ become highly nonlinear. But in a weak field approximation such model could reproduce a correct kinetic term for the vector field. Note also that the corrections to the $R_{\mu\nu}$ tensor here start with the terms quadratic in $F_{\mu\nu}$ and there is no term linear in it in contrast with [19].

5. Conclusion

In this paper we have shown that there indeed exists a dual formulation of gravity in terms of Lorentz connection $\omega_\mu{}^{ab}$ field. Such formulation turns out to be highly non-linear higher derivatives theory, so it is not an easy task (if at all possible) to give compact formulation at full non-linear level. However it is possible to construct such theory iteratively, order by order in fields as we have done in the linear approximation here. Also we have shown that the so called exotic parity-violating interactions for massless spin-2 particles could be considered just as such dual formulation of usual gravitational interactions.

Also we have shown that the dualization procedure based on the use of (Anti) de Sitter background space could be applied to the gravity theory in a Palatini formalism and leads to the formulation in terms of affine connection. In this, the final Lagrangian coincides with that of Eddington [13]. A number of interesting question arises, for example, whose related with the gauge symmetries of such formulation, which deserve further study.

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